Journal of Statistical Physics, Vol. 58, Nos. 1/2, 1990

The Scale Equations in the Critical Dynamics of Fluctuating Systems

Yu. M. Ivanchenko,[1] A. A. Lisyanskii,[1] and A. E. Filippov[1]

Received December 23, 1988

The scale equation method is applied to the investigation of the critical dynamics of systems described by Ginzburg–Landau functionals of the most general form. The method does not require renormalizability of the Ginzburg–Landau functional and does not make use of the scaling invariance hypothesis.

KEY WORDS: Critical dynamics; fluctuations; renormalization group; phase transition; scale equations.

1. INTRODUCTION

The renormalization group (RG) method made it possible, at least in principle, to solve the phase transition problem. This method allowed one not only to understand the physics of the phenomena occurring in the critical region and to predict quite a number of new effects, but also to calculate critical asymptotics with an exceptionally high accuracy.[1-5] Both approaches using the perturbation theory in the φ^4 model[1-3] and approaches based on other approximation schemes[4,5] for the investigation of the exact RG equations proposed by Wilson[6] appeared to be rather effective.

We have developed[7] a new approach to investigate the critical state thermodynamics. This approach is based on the investigation of the exact equations for the correlation functions, which were called the scale equations (SE) (this term is determined by the method of the derivation of the SE and is not in any way connected with the scale invariance hypothesis). The SE technique is applicable to the investigation of systems of the most

[1] Donetsk Physico-Technical Institute of the Ukrainian Academy of Sciences, 340114 Donetsk, USSR.

0022-4715/90/0100-0295$06.00/0

general form described by Ginzburg–Landau functionals containing anharmonics of all orders (i.e., this method does not require the Ginzburg–Landau functional to be renormalizable, as opposed to the field theory approach to the phase transition problem[8]). Moreover, this method does not use the hypothesis on the scale invariance which is the basic hypothesis of the Wilson theory. Nevertheless, the SE method is closely connected with the RG approach; the exact RG equations in the form close to the initial formulation by Wilson follow from the SE (in this case the procedure of the derivation of the RG equations in the new approach appears to be very simple). The use of the SE makes it possible to solve some mathematical problems which arise in the conventional approaches (for example, the problem of the elimination of the so-called "redundant operators"). Finally, using the SE, one can obtain results which the conventional approach cannot give (e.g., one can obtain the exact momentum dependence of the correlation function at the critical point over the whole range of the momentum change).

The present paper is aimed at generalizing the SE method to investigate the critical dynamics. It will be shown that using the SE method, one can obtain all the results given by the standard approach. This method, as in the static case, uses the Ginzburg–Landau functional of the most general form with nonlocal vertices. Moreover, in the calculations up to the critical exponents one may leave unspecified the relaxation equation for the order parameter, i.e., it is possible not to set the explicit form of the momentum dependence of the function $\Gamma(q^2)$ in the time-dependent Ginzburg–Landau model (see refs. 9 and 10).

Finally, the quantum generalization of the classical Ginzburg–Landau functional used in this approach can be derived from the microscopic quantum Hamiltonian (see, for example, refs. 11–16, and Appendix D, where the Ginzburg–Landau functional for the liquid–vapor transition is obtained from the Hamiltonian of the interacting Bose gas). In this sense, this approach can be regarded as a first-principles one.

The paper is set on a physically rigorous level and is organized as follows. In Section 2 the SE are derived for the arbitrary transformation-invariant system whose Ginzburg–Landau functional contains all even powers of the vector field. Section 3 presents the method for the elimination of the redundant operators generating the determination of the Fisher exponent η. Section 4 is intended to calculate values η and Γ within the ε expansion and to prove the equivalence of the approach developed here to the standard theory for the φ^4 model. The solution used here for the RG equation fixed point is given in Appendix A. Appendix B deals with the additional investigation of the problem of the interconnection between the RG equations obtained in various approaches. Appendix C considers the problem of the universality of the obtained critical asymptotics. Finally, independent interest exists in deriving, in as compact a form as possible, the Ginzburg–Landau quantum functional from the microtheory. This is done in Appendix D, where this derivation is performed for a system of interacting bosons.

2. DERIVATION OF THE SCALE EQUATIONS

Let us start from the Ginzburg–Landau functional of the most general form, which possesses all even[2] powers of the n-component vector field φ. We shall also take into consideration the fact that the functional vertices may depend both on momenta and on frequencies. Although we shall consider the classical system, to make the following calculations it is convenient to use the Ginzburg–Landau functional for the quantum systems and to pass to the classical limit at the end of the calculations. The Ginzburg–Landau functional for the translation-invariant system can be written as

$$\beta\mathscr{H}_I[\varphi]=\sum_{k=1}^{\infty} 2^{1-2k}\sum_{\{\omega_i\}}\int_{\{q_i\}} g_k^{\alpha_1\cdots\alpha_{2k}}\{q_i;\omega_i\} \times(2\pi)^d\,\delta\left(\sum_{i=1}^{2k} q_i\right)\prod_{i=1}^{2k}\varphi^{\alpha_i}(q_i,\omega_i)\,\delta_{\sum_{i=1}^{2k}\omega_i,0} \tag{2.1}$$

Here $\beta=1/T$ is the inverse temperature, q_i are the d-dimensional momenta, $\omega_i=2k\pi iT$, $k=0,\ \pm 1,\ \pm 2,...$, α_i are the vector indices, $\int_q=V^{-1}\Sigma_q$, and the Fourier transformations are determined by the relationship $\varphi(l)=\int_q\varphi(q)\exp(iql)$. Over the repeating spatial indices (vector indices) an integration (summation) is supposed to be performed.

The integrals over the momenta appearing in the theory should be cut off at some momentum Λ. To avoid the introduction of a cutoff momentum into the concrete integrals, one can use the quantum field theory procedure of singular function regularization.[17] For this purpose the full Ginzburg–Landau functional can be represented as a sum of functional (2.1) and a term quadratic over φ

$$\begin{aligned}\beta\mathscr{H}_0[\varphi]&=\frac{1}{2}\sum_{\omega}\int_q G_0^{-1}(q,\omega)\,|\varphi(q,\omega)|^2\\&=\frac{1}{2}\sum_{\omega}\int_q f(q^2,\omega)\,S^{-1}\left(\frac{q^2}{\Lambda^2}\right)|\varphi(q,\omega)|^2\end{aligned} \tag{2.2}$$

[2] The following consideration can be easily generalized to the most general functional also containing odd powers of φ.

where $f(q^2, \omega)$ is a smooth function degenerate in the static limit into $f(q^2, 0) = q^2$ and the monotonic function $S(q^2/\Lambda^2)$ provides the momentum integral cutoff if $S(x \to 0) \to 1$ and $\lim_{x \to \infty} [S(x)\, x^m] = 0$ at arbitrary m. It should be noted that this procedure can provide both smooth and abrupt cutoff. In the latter case $S(x) = \theta(1-x)$, where $\theta(x)$ is the stepped function.

The particular form of the function $f(q^2, \omega)$ is determined by the structure of the model under investigation. It can be determined within the phenomenological approach when the Ginzburg–Landau functional is supplemented with a kinetic equation for $\varphi_q(t)$. An alternative approach is possible where the function $f(q, \omega)$ is determined from the microscopic Hamiltonian of the system (an example of this calculation is given in Appendix D). The following forms are the most widespread:

$$f(q, \omega) = q^2 + \omega \Gamma^{-1} \Lambda^2 \tag{2.3a}$$

$$f(q, \omega) = q^2 + \omega \Lambda^4 / \Gamma q^2 \tag{2.3b}$$

The technique of the following calculations is organized so that the particular form of the function $f(q^2, \omega)$ is nonessential until the point where the choice of this particular form is particularly specified. We explicitly isolated the dimensional parameter into combinations like ω/Γ, having assumed this parameter to coincide with the square of the cutoff momentum Λ. In fact, any functional of the type (2.1) can be obtained on the basis of the microscopic Hamiltonian of the system. The natural parameter of the momentum integral cutoff is represented by the value $\sim 1/a$, where a is the atomic dimension (in the solid state case a is the crystal lattice constant; or the thermal length as in Appendix D). On the other hand, the parameter a makes all functions of momenta in combination $[qa] \sim 1$ dimensionless. Finally, in view of the fact that $\mathscr{H}$ has the dimension of energy, the function $f(q^2, \omega)$ can be rewritten in the most general form as

$$f(q^2, \omega) = \Lambda^2 \phi(q^2/\Lambda^2, \omega/\Gamma) \tag{2.4}$$

It is essential that at the Kadanoff-type transformations the magnitude of q (and, consequently, of Λ, too) is transformed simultaneously with the scale changes.

Let us determine now the mth-order correlation function

$$\left\langle \prod_{i=1}^{m} \varphi^{\alpha_i}(q_i, \omega_i) \right\rangle = (2\pi)^d\, \delta\left(\sum_{i=1}^{m} q_i \right) \delta_{\sum_i^m \omega_i, 0} A_m^{\alpha_1 \cdots \alpha_m}\{q_i; \omega_i\} \tag{2.5}$$

This function is linearly connected with the Green temperature function of the initial quantum Hamiltonian (see, for example, Appendix D). At the end of the calculations one should perform an analytical continuation of the function $\hat{A}_m\{q_i; \omega_i\}$ from the discrete frequencies to the real axis. Then this function will determine the physical response function. $\hat{A}_m$ is a function of q_i, ω_i of the cutoff momentum Λ, and all vertices $\hat{g}_k$. To derive the SE for the function $\hat{A}_m$ one should find its change at the scale transformations, i.e., at the transformations of the $q_i = \bar{q}_i\lambda$, $\Lambda = \bar{\Lambda}\lambda$ type. The change in the cutoff momentum affects the structure of the integrals involved in the definition of the average $A_m^{\alpha_1 \cdots \alpha_m}$. For the function $\hat{A}_m$ to preserve its initial form, one should also change field variables $\varphi(q\lambda, \omega) = \lambda^{-\Delta_\varphi}\bar{\varphi}(\bar{q}, \omega)$. It should be noted that the Λ change does not touch the integrals over ω and, consequently, mute variables ω, on which values φ and $\hat{g}_k$ depend, are not transformed.

The choice of Δ_φ is quite arbitrary. Let us use Δ_φ in the simplest form $\Delta_\varphi = (d+2)/2$. This form keeps the functional $\mathscr{H}_0$ unchanged in new variables (at the corresponding change of Γ^{-1}). Rather simple transformations yield

$$\delta_{\sum_{i=1}^m \omega_i, 0}\,\delta\left(\sum_{i=1}^m q_i\right) \hat{A}_m(\{q_i; \omega_i\}, \Lambda, \hat{g}_k\{q_i\})$$

$$= \lambda^{m(d+2)/2}\delta\left(\lambda \sum_{i=1}^m q_i\right) \delta_{\sum_i^m \omega_i, 0}\hat{A}_m(\{\lambda q_i; \omega_i\}, \Lambda\lambda, \lambda^{\varepsilon_k}\hat{g}_k\{\lambda q_i, \omega_i\}) \tag{2.6}$$

where $\varepsilon_k = d + k(2-d)/2$. Differentiating the relationship (2.6) with respect to λ and setting $\lambda = 1$, one obtains

$$\left\{m\frac{d+2}{2} - d - dV\frac{\partial}{\partial V} + \sum_{i=1}^m q_i \frac{\partial}{\partial q_i} + \Lambda\frac{\partial}{\partial \Lambda}\right.$$
$$\left. + \sum_{k=1}^\infty \int_{\{q_i\}} \left[\left(\varepsilon_k + \sum_{i=1}^{2k} q_i \frac{\partial}{\partial q_i}\right) \hat{g}_k\{q_i; \omega_i\}\right] \frac{\delta}{\delta \hat{g}_k}\right\}$$
$$\times \hat{A}_k(\{q_i; \omega_i\}, \Lambda, \hat{g}_k\{q_i\lambda; \omega_i\}) = 0 \tag{2.7}$$

The operator $-dV\,\partial/\partial V$ appears when the function $\hat{A}_m$ explicitly depends on the volume V.

Taking into account the explicit form of the functions $f(q^2, \omega)$ [e.g., (2.3a) and (2.3b)], one can see that in the general case the operator $\Lambda\,\partial/\partial\Lambda$ contains two terms. The first term is connected with the explicit dependence of $f(q^2, \omega)$ on Λ^2 and can be rewritten as $\Delta_\omega \omega\,\partial/\partial\omega$ [$\Delta_\omega = 1/2$ for the case of (2.3a) and $\Delta_\omega = 1/4$ for (2.3b)], and the second term (it is this term that will be denoted by $\Lambda\,\partial/\partial\Lambda$) is due to the presence of the momentum integral cutoff. For discrete Matsubara frequencies the operator $\omega\,\partial/\partial\omega$ has quite a formal meaning. However, at the end of the calculations when the functions (2.5) are analytically extended from the discrete frequencies to the real axis this operator becomes correctly determined.

The critical asymptotics of thermodynamic functions do not depend on the value of Λ; therefore, one can, quite naturally, believe that functions $\hat{A}_m$ depend on Λ only as a parameter. In this case the derivative $\Lambda\, \partial/\partial\Lambda$ can be expressed by some operator depending on other variables. For this purpose let us find the change in $\hat{A}_m$ at a small variation of Λ. First it should be noted that the averaging over a Gaussian field φ (denoted by $\langle\cdots\rangle_0$) can be replaced with two independent averagings over the Gaussian fields φ_1 and φ_2 so that $\varphi=\varphi_1+\varphi_2$ and the sum of their correlators G_{01} and G_{02} is equal to the initial field correlator. Let us consider now the change in the average of some functional $W[\varphi]$ over the Gaussian field φ with Hamiltonian (2.2). We divide φ into a sum $\varphi_1+\varphi_2$ so as to make the φ_1 field correlator equal to

$$G_{01}(q, \Lambda)=\langle\varphi_1(q, \omega)\,\varphi_1(-q, -\omega)\rangle=G_0(q, \omega; (1-\delta)\Lambda)$$

Correlator G_{02} acquires the form

$$\begin{aligned} G_{02}(q, \omega; \Lambda) &= G_0(q, \omega; \Lambda)-G_0(q, \omega; (1-\delta)\Lambda) \\ &= \delta\cdot\Lambda\,\frac{\partial G_0}{\partial\Lambda}\equiv 2\delta\cdot h(q, \omega) \end{aligned}$$

Since $\langle W\rangle_0=\langle W\rangle_{0,12}$ and the function G_{02} is small due to the small value of δ, the integration over φ_2 is easily performed to yield

$$\langle W\rangle_{0,\Lambda}=\langle(1+\delta\cdot\hat{L})\,W\rangle_{0,\Lambda(1-\delta)}$$

where the operator $\hat{L}$ has the form

$$\begin{aligned} \hat{L} &= T\sum_\omega\int_q h(q, \omega)\,\frac{\delta^2}{\delta\varphi(q, \omega)\,\delta\varphi(-q, -\omega)} \\ \Lambda\,\frac{d}{d\Lambda}\,\langle W\rangle_0 &= \langle\hat{L}W\rangle_0 \end{aligned} \tag{2.8}$$

Applying relationship (2.8) to the averages over the total Hamiltonian, one gets

$$\begin{aligned} \Lambda\,\frac{d}{d\Lambda}\,\langle W\rangle &= \frac{[\langle\hat{L}e^{-\mathscr{H}_I}W\rangle_0-\langle W\rangle\langle\hat{L}e^{-\mathscr{H}_I}\rangle_0]}{\langle e^{-\mathscr{H}_I}\rangle} \\ &= T\sum_\omega\int_q h(q, \omega)\left\langle\frac{\delta^2\mathscr{H}_I}{\delta\varphi(q, \omega)\,\delta\varphi(-q, -\omega)}-\frac{\delta\mathscr{H}_I}{\delta\varphi(q, \omega)}\,\frac{\delta\mathscr{H}_I}{\delta\varphi(-q, -\omega)}\right| W\right\rangle \\ &\quad+\langle\hat{L}W\rangle-2T\sum_\omega\int_q h(q, \omega)\left\langle\frac{\delta\mathscr{H}_I}{\delta\varphi(q, \omega)}\,\frac{\delta W}{\delta\varphi(-q, -\omega)}\right\rangle \end{aligned}$$

where $\langle \cdots | \cdots \rangle$ denotes the connected average. Knowing that

$$\left\langle \frac{\delta \mathscr{H}_I}{\delta\varphi(q,\omega)} \frac{\delta W}{\delta\varphi(-q,-\omega)} \right\rangle = \left\langle \frac{\delta^2 W}{\delta\varphi(q,\omega)\,\delta\varphi(-q,-\omega)} \right\rangle - G_0^{-1} \left\langle \varphi(q,\omega) \frac{\delta W}{\delta\varphi(-q,-\omega)} \right\rangle$$

and using the relationship (2.7) (after rather simple though tedious calculations), one gets the final expression for the desired SE

$$\left\{ m\frac{d+2}{2} - d - dV\frac{\partial}{\partial V} + \sum_{i=1}^{m} \left[q_i \frac{\partial}{\partial q_i} + \Delta_{\omega_i}\omega_i \frac{\partial}{\partial \omega_i} + 2G_0^{-1}h + \hat{\phi}[\hat{g}_k] \right] \right\} \times \delta_{\sum_i \omega_i} \delta\left(\sum_i q_i\right) \hat{A}_m\{q_i\} = \sum_{j=1}^{m} \sum_{l\neq j}^{m} \delta_{\omega_j,-\omega_l} \delta(q_j+q_l)\, h(q_j,\omega_j)\, \delta_{\alpha_j\alpha_l} \left\langle \prod_{i\neq j\neq l}^{m} \varphi^{\alpha_i}(q_i,\omega_i) \right\rangle$$

Here the operator $\hat{\phi}$ is determined by

$$\hat{\phi}[\hat{g}_k] = T \sum_{k=1}^{\infty} \sum_{\omega_i} \int_{\{q_j\}} \hat{U}_k[\hat{g}_k\{q_i,\omega_i\}] \frac{\delta}{\delta \hat{g}_k} \tag{2.9}$$

$$\begin{aligned} U_k^{\alpha_1\cdots\alpha_k}[q_i] &= \left(\varepsilon_k + \sum_{i=1}^{2k} q_i \frac{\partial}{\partial q_i} \right) + T \sum_{\omega,\omega'} \int_{qq'} (2\pi)^d \\ &\times \delta(q+q')\, h(q,\omega)\, \delta_{\omega,-\omega'} \Big\{ (k+1)(2+\ +1)\, g_{k+1}^{\gamma\gamma\alpha_1\cdots\alpha_{2k}}(q,q',\{q_i,\omega_i\}) \\ &- \sum_{m=1}^{k+1} 2m(k-m+1)\, \hat{S}[g_m^{\gamma\alpha_1,\ldots,\alpha_{2m-1}}(q,q_1,\ldots,q_{2m-1};\omega,\omega_1,\ldots,\omega_{2m-1}) \\ &\times g_{k-m+1}^{\gamma\alpha_{2m}\cdots\alpha_{2k}}(q',q_{2m},\ldots,q_{2k};\omega',\omega_{2m},\ldots,\omega_{2k})] \Big\} \end{aligned} \tag{2.10}$$

where the operator $\hat{S}$ performs the symmetrization of the product $\hat{g}_m \times \hat{g}_{k-m+1}$ with respect to permutation of variables q_i, ω_i and q_j, ω_j (with simultaneous permutation of indices). In relationships (2.8) and (2.9) the value λ is assumed to be equal to Λ^{-1}, so the momenta are measured in units of Λ and the cutoff parameter is equal to unity. Note that on the right-hand side of Eq. (2.8) there is a function $\hat{A}_{m-2}$ which has no momenta $q_{j,l}$ and frequencies $\omega_{j,l}$.

To conclude this section, it should be mentioned that the SE are closely connected with the exact (functional) RG equations.[6] Namely, the characteristic system of the operator $\hat{\phi}$ coincides with the RG equations in the differential form

$$\dot{\hat{g}}_k\{q_i, \omega_i\} = \hat{U}_k[\hat{g}_j\{q_i, \omega_i\}]; \qquad k = 1, 2,... \tag{2.11}$$

These equations do not contain derivatives with respect to ω_i and, as can be directly seen (from Appendix A) they lead to fixed points for vertices $\hat{g}_k$ coinciding with the analogous quantities in statics.

The right-hand side of the system of equations (2.11) contains operators $\hat{U}_k[\hat{g}_j]$ determined by means of Eq. (2.10). Direct substitution of functional $\mathscr{H}_I$ of (2.1) demonstrates that the system (2.11) is generated by the equation

$$\dot{\mathscr{H}}_I = dV\frac{\partial \mathscr{H}_I}{\partial V} + T\sum_{\omega}\int_q \left\{\left[\frac{d+2}{2}\varphi(q,\omega) + q\frac{\partial}{\partial q}\varphi(q,\omega)\right]\frac{\delta \mathscr{H}_I}{\delta\varphi(q,\omega)} + \dot{h}(q,\omega)\left[\frac{\delta^2\mathscr{H}_I}{\delta\varphi(q,\omega)\,\delta\varphi(-q,-\omega)} - \frac{\delta\mathscr{H}_I}{\delta\varphi(q,\omega)}\frac{\delta\mathscr{H}_I}{\delta\varphi(-q,-\omega)}\right]\right\} \tag{2.12}$$

Equation (2.12) is considerably more compact and allows one to reproduce quite easily the equations for all $\dot{\hat{g}}_k$.

3. ELIMINATION OF REDUNDANT OPERATORS

In the investigation of the exact RG equations the "redundant operators" problem appears.[18] Thus, for instance, the transformation of variables $\varphi \to \varphi/\xi$ in the Hamiltonian should not lead to a new critical behavior. At the same time this transformation results in a new value of fixed points of the RG equations. The SE provide quite new possibilities in view of the elimination of redundant operators (since these SE are written for the physical functions, but not for the functionals as is the case with the RG ones). This is due to the fact that in deriving the SE one can take into account not only the invariance of thermodynamic functions with respect to the transformation (2.6), but with respect to other possible transformations, too. This sharply reduces the class of the SE solutions and eliminates the operators associated with the transformations taken into account. Since the problem of redundant operators appears already in the static case and the result of its solution will be used in the description of dynamics, let us consider in the beginning the purely static problem.

For the correlation function $\hat{A}_m$ one can write down a more general scale equation than (2.6), i.e.,

$$\hat{A}_m(\{q_i, \omega_i\}, \Lambda, \hat{g}_k\{q_i, \omega_i\}) = \left[\prod_{i=1}^{m} c(q_i\lambda)\right] \hat{A}_m(\{\lambda q_i, \omega_i\}, \Lambda\lambda, \hat{\hat{g}}_k\{q_i\lambda, \omega_i\}) \tag{3.1}$$

where the vertices $\hat{\hat{g}}_k$ are connected with the initial relations

$$\sum_{r\kappa_r} \hat{g}_{kr\kappa_r}\left[\prod_{i=1}^{2k} c(q_i\lambda)\right] p_{r\kappa_r}\{q_i\}\, \delta\left(\sum_{i=1}^{2k} q_i\right) = \sum_{r\kappa_r} \hat{\hat{g}}_{kr\kappa_r} p_{r\kappa_r}\{q_i\}\, \delta\left(\sum_{i=1}^{2k} q_i\right)$$

$\{p_{r\kappa_r}\}$ is the set of $\{\kappa_r\}$ homogeneous polynomials of order of r composed of vectors q_i.

The latter procedure is equivalent to the change in the scale dimensionality of field variables Δ_φ. Previously $\Delta_\varphi = (d+2)/2$ was chosen so that the functional $\mathscr{H}_0$ was unchanged in new variables [so that $c(q, \lambda)$ was constant, $c = \lambda^{(d+2)/2}$], whereas now $\Delta_\varphi = [d+2-\eta(q)]/2$ [and consequently $c(q, \lambda) = \lambda^{[d+2-\eta(q)]/2}$] where $\eta(q)$ is quite an arbitrary function. We shall make use of this arbitrariness below. Knowing that $\Delta_\varphi \neq (d+2)/2$ and substituting the change of functional $\mathscr{H}_0$ at taking account of $\eta(q) \neq 0$ to take the derivative $\dot{\mathscr{H}}_I$, one can replace an equation of the type (2.12) with the more generalized expression

$$\begin{aligned}\dot{\mathscr{H}}_I = {} & dV\frac{\partial \mathscr{H}_I}{\partial V} + \frac{1}{2}\int_q \eta(q) - \frac{1}{2}\int_q \eta(q)\, G_0^{-1}(q)\, |\varphi(q)|^2 \\ & + \int_q \left\{\left[\frac{d+2-\eta(q)}{2}\varphi(q) + q\frac{\partial}{\partial q}\varphi(q)\right]\frac{\delta \mathscr{H}_I}{\delta\varphi(q)}\right. \\ & \left. + h(q)\left[\frac{\delta^2 \mathscr{H}_I}{\delta\varphi(q)\,\delta\varphi(-q)} - \frac{\delta \mathscr{H}_I}{\delta\varphi(q)}\frac{\delta \mathscr{H}_I}{\delta\varphi(-q)}\right]\right\}\end{aligned} \tag{3.2}$$

In the static case the functional $\mathscr{H}_0$ at φ^2 has a function $G_0^{-1}(q)$ which possesses all powers of q starting from the second one. On the other hand, the functional $\mathscr{H}_I$ at φ^2 also has the function of momentum $g_1(q)$. The transformation (3.1) allows us to avoid this arbitrariness. For this purpose we use Eq. (3.2) to write the equation for the derivative $\dot{g}_1(q)$,

$$\dot{g}_1(q) = [2-\eta(q)]\, g_1(q) + D(q) - G_0^{-1}(q)\,\eta(q) \tag{3.3}$$

$$D(q) = Q_1(q) - g_{10}^2 h(q) \tag{3.4}$$

$$Q_k(q_i) = (k+1) \int_p h(p) \left[\frac{n}{2} g_{k+1}(-p, p; q_1, q_{\bar{1}}, \ldots) + k\hat{S} g_{k+1}(p, q_1; -p, q_{\bar{1}}; q_2, q_{\bar{2}}, \ldots, q_k, q_{\bar{k}}) \right] \tag{3.5}$$

Here the isotropic model is used, where the vertex g_k depends on $2k$ momenta $(q_1, q_{\bar{1}}, \ldots, q_k, q_{\bar{k}})$ and it remains invariant at any permutations of variable pairs q_i, $q_{\bar{i}}$ and q_j, $q_{\bar{j}}$ between each other and within each pair. Moreover, the function Q_k as well as the vertices g_k are proportional to δ-functions $\delta(q_1 + \cdots + q_{\bar{k}})$; therefore the function $Q_1(q)$ depends only on one momentum.

Let us choose $\eta(q)$ so as to get $\dot{g}_1(q \neq 0) = 0$. Thus, if the trial value is $g_1(q \neq 0) = 0$, this condition will be also satisfied at each stage of the RG transformation, i.e., $g_1 = g_1(q = 0) = g_{10}$. Using Eq. (3.3), one can easily get for $\eta(q)$ an explicit expression

$$\eta(q) = \eta(0) + \frac{[D(q) - D(0)] - \eta(0)\, G_0^{-1}(q)}{G_0^{-1}(q) + g_{10}} \tag{3.6}$$

where

$$\eta(0) = \frac{1}{2d} \frac{d^2 D(q)}{dq^2} \bigg|_{q=0} \tag{3.7}$$

The function $\eta(q)$, as will be shown below, is closely connected with the Fisher exponent η. It should be noted, first, that Eq. (3.2) gives rise to a new operator $\hat{\hat{\Phi}}$ determined via $\hat{U}_k$ using Eq. (2.11). Moreover, the function $\eta(q)$ will be explicitly included in the scale equation for the correlation function $\hat{A}_m$ by means of the combination $\sum_{i=1}^m [d + 2 - \eta(q_i)]/2$, so that the corresponding equation will acquire the form

$$\left\{ m\frac{d+2}{2} - d - dV\frac{\partial}{\partial V} + \sum_{i=1}^{m} \left[-\frac{\eta(q_i)}{2} + q_i \frac{\partial}{\partial q_i} + 2G_0^{-1}(q_i)\, h(q_i) \right] + \hat{\hat{\phi}} \right\}$$
$$\times \delta\left(\sum_{i=1}^{m} q_i \right) \hat{A}_m\{q_{1,r>0} = 0\}$$
$$= \sum_{j \neq l = 1}^{m} \delta(q_l + q_j)\, h(q_j)\, \delta_{\alpha_j \alpha_l} \left\langle \prod_{i \neq j, l}^{m} \varphi^{\alpha_i}(q_i) \right\rangle$$

To clarify the physical meaning of $\eta(q)$, it is quite enough to have a static scale equation for the two-point correlation function

$$\delta_{\alpha\beta} G(q) = \frac{1}{V} \langle \varphi^\alpha(q)\, \varphi^\beta(-q) \rangle = A_2^{\alpha\beta}(q, -q)$$

In accordance with the scaling hypothesis, the functional $\mathscr{H}_I$ reaches its stable fixed value at the critical point and, consequently, $\dot{g}_k = \hat{U}_k = 0$. The operator $\hat{\phi}$ also becomes zero and the equation for $G(q)$ has the form

$$\left[2 - \eta(q) + 4q^2 h(q)\, S^{-1}(q^2) + q\frac{\partial}{\partial q}\right] G(q) = 2h(q) \tag{3.8}$$

This equation should be solved at the initial condition $G(q \to \infty) \to G_0(q)$. A direct check shows that the solution of Eq. (3.8) has the form

$$G(q) = \frac{S(q^2)}{q^2}\, e^{-\kappa(q^2)} \left\{1 - S(q^2) \int_{q^2}^{\infty} dt \frac{dS^{-1}}{dt}\, [e^{\kappa(t)} - 1]\right\} \tag{3.9}$$

where

$$\kappa(q^2, t) = \frac{1}{2}\int_t^{q^2} d\tau \frac{\eta(\tau)}{\tau}$$

and taking into account that in the static limit $G_0(q) = q^{-2}S(q^2)$, so $h(q^2) = -dS(q^2)/dq^2$. Being interested in the asymptotic behavior of $G(q)$ at $q \to 0$, one obtains

$$\begin{aligned} G(q) &= q^{-2+\eta(0)} \left\{\exp\left[-\kappa(q^2) - \frac{\eta(0)}{2}\ln q^2\right]\right. \\ &\quad \times \left.\left[1 - \int_{q^2}^{\infty} dt \frac{dS^{-1}}{dt}\, (e^{\kappa(t)} - 1)\right]\right\} \\ &\equiv B(q)\, q^{-2+\eta(0)} \end{aligned} \tag{3.10}$$

where $B(q)$ has a finite limit at $q \to 0$.

In principle, Eq. (3.9) sets the behavior $G(q)$ at the critical point at arbitrary values of q if the function $\eta(q)$ is calculated (e.g., within the perturbation theory) or is somehow approximated. However, within the scope of this paper it is quite sufficient to state that the limit $\eta(0)$ of this function coincides with the Fisher exponent η as is evidenced from (3.11).

Now let us come back to the dynamic problem. In addition to the wave vector dependence, G_0^{-1} is also a function of frequency ω and the value Γ. Upon renormalization the generation of the vertex $g_1(q, \omega)$ takes place, which now depends on q and ω. Proceeding as before, one can eliminate the dependence of g_1 on q. Using the generalized field $\varphi(q, \omega)$ transformation with the function $\eta(q, \omega)$, one obtains

$$\dot{g}_1(q, \omega) = [2 - \eta(q, \omega)]\, g_1 + D(q, \omega) - G_0^{-1}(q, \omega)\, \eta(q, \omega)$$

This obviously yields equations for $\eta(q, \omega)$ similar to (3.6), (3.7) [see also Eqs. (4.5)–(4.6) of the following section]. In this case, of course, all functions on the right-hand sides of these equations [and consequently $\eta(q, \omega)$] depend on ω. In view of the correspondence to the static limit, the Fisher exponent $\eta = \eta(0, 0)$ should be determined now by the formula

$$\eta(0, 0) = \frac{1}{2d} \frac{d^2 D(q, \omega)}{dq^2}\bigg|_{q=\omega=0}$$

Finally, note that in the determination of values Q_k [Eq. (3.5)] there is the replacement $\int_p \to T \sum_\omega \int_p$. Due to this, the coincidence of the value $\lim_{\omega \to 0} \eta(q, \omega)$ with the static function $\eta(q)$ becomes less evident. This coincidence will be proved in the next section. For a fixed choice of $\eta(q, \omega)$ the dependence of $\dot{g}_1$ on ω persists. The only way to compensate this dependence is to make the replacement $\Gamma \to \Gamma(\omega)$. In other words, Γ becomes a "charge" and is renormalized. This is in agreement with a similar result obtained previously.[9,10]

4. CALCULATION OF η AND Γ IN THE ϵ EXPANSION. INTERCONNECTION OF VARIOUS APPROACHES

Equations (2.11) together with definitions (3.5)–(3.7) [taking account of the replacements $h(q) \to h(q, \omega)$ and $\int_q \to \sum_\omega \int_q$] determine $\eta(q, \omega)$ and $\Gamma(\omega)$ formally exactly. In principle, they permit alternatives to obtaining the solutions both on the basis of perturbation theory (over small vertices, $\varepsilon = 4 - d$, and over inverse component number $1/n$) or without it (on other grounds). In any case, however, at a certain stage of investigation it will be necessary to solve these equations within the approximations of other approaches (e.g., ε expansion) and compare the results. At this stage it is of interest not only to show numerical closeness or coincidence of the sought values (Fisher exponent η, dynamic exponent z, and so on), but also to make sure that there is an analytical coincidence of integrals in the results or even an interrelation between the RG equations in various approaches. This problem will be investigated below and in the Appendices.

The expansion parameter will be represented by the quantity $\varepsilon = 4 - d$ introduced in the pioneering work by Wilson and Fisher[19] and widely used now in the theory of critical phenomena. This parameter was also used in fundamental work on fluctuating system dynamics in the conventional approach.[9,10,20] Within the framework of the new scheme presented here the RG equations for the fixed point are written in Appendix A and solved with an accuracy of the second order in ε. As a result, Eq. (A15) is obtained for the ω- and q-dependent contribution to the derivative $\dot{g}_1(q, \omega)$. The contribution can be conveniently written as

$$D(q, \omega) = 24(n+2)\, g_{20}^2 [I(q, \omega) - I(k; q=0, \omega)] \tag{4.1}$$

where the integral $I(q, \omega)$ is determined by the equation

$$I(q, \omega) = T^2 \sum_{\omega_{1,2}} \int_{k,p} h(p, \omega_1)\, h(k, \omega_2) \times \iint_0^1 \frac{dx}{x}\, dy\, yh([p + (k+q)x]\, y, -\omega - \omega_1 - \omega_2) \tag{4.2}$$

The quantity g_{20} at the fixed point is specified by relationship (A12),

$$g_{20} = \frac{1}{4(n+8)\,\Psi} \tag{4.3}$$

which also possesses an integral of the function $h(q, \omega)$ in the form

$$\Psi = T \sum_{\omega} \int_p h(p, \omega) \int_0^1 dy\, yh(py, \omega) \tag{4.4}$$

Rewriting the equation for $\dot{g}_1(q, \omega)$ in the form of

$$\dot{g}_1(q, \omega) = [2 - \eta(0, 0)]\, g_{10} + D(0, 0) + [D(q, \omega) - D(0, \omega)] + [D(0, \omega) - D(0, 0)] - G_0^{-1}(q, \omega)[\eta(q, \omega) - \eta(0, 0)] \tag{4.5}$$

and choosing the function $\eta(q, \omega)$ in accordance with Eqs. (3.6) and (3.7),

$$\eta(q, \omega) = \eta(0, 0) - \frac{[D(q, \omega) - D(0, \omega)] - \eta(0, 0)\, G_0^{-1}(q, \omega)}{g_{10} + G_0^{-1}(q, \omega)} \tag{4.6}$$

then at $\eta(0, 0) = (2d)^{-1}\, d^2D/dq^2\,|_{q=\omega=0}$ one gets

$$\dot{\Gamma}^{-1} = \lim_{\omega \to 0} \frac{D(0, \omega) - D(0, 0)}{\omega} = \left.\frac{dD}{d\omega}\right|_{q=\omega=0} \tag{4.7}$$

Thus, our task is reduced to the analysis of integrals $I(q, \omega)$ and Ψ involved in the function $D(q, \omega)$. Let us mention here one more useful property of the function $h(q, \omega)$. In accordance with the definition, one can write

$$h(q, \omega) = \Lambda^2 \left.\frac{\partial G_0(q, \omega)}{\partial \Lambda^2}\right|_{\Lambda=1} \tag{4.8}$$

On the other hand, $G_0^{-1}(q, \omega) = f(q^2, \omega)\, S^{-1}(q^2/\Lambda^2)$, so, taking the general form (2.4) of the function f into account, we obtain

$$G_0(q, \omega) = \Lambda^{-2} \phi^{-1}(q^2/\Lambda^2, \omega/\Gamma)\, S(q^2/\Lambda^2) \tag{4.9}$$

Substituting (4.9) into (4.8), one can see that $h(q, \omega)$ satisfies the equality

$$h(q, \omega) = -\frac{\partial}{\partial q^2} [q^2 G_0(q, \omega)] \Big|_{\Lambda = 1} \tag{4.10}$$

Let us consider now the value of Ψ. Using its definition (4.4) and Eq. (4.10), one obtains

$$\begin{aligned} \Psi &= \frac{T}{2} \sum_\omega \int_p h(p, \omega)\, p^{-2} \int_0^p dp^2\, h(p, \omega) \\ &= -\frac{T}{2} \sum_\omega \int_p h(p, \omega)\, G_0(p, \omega) \\ &= \frac{T}{4} \sum_\omega \int_p G_0^2(p, \omega) \Big|_{\Lambda = 1} \end{aligned}$$

The particular structure of the functions $f(q^2, \omega)$ [see (2.3a), (2.3b)], $f(q^2, \omega) = q^2 + \omega \varphi(\Gamma, \Lambda, q^2)$ is such that one can make the summation over $\omega_k = 2\pi k i T$ explicitly,

$$T \sum_\omega G_0^2(p, \omega) = S^2(p^2)\, T \sum_\omega \frac{1}{(p^2 + \omega\varphi)^2} = S^2 \varphi^2 \frac{\partial v(x)}{\partial x} \Big|_{x = -p^2}$$

where $v(\omega) = (e^{\beta\omega} - 1)^{-1}$. Since further on we shall be interested in finite temperatures and low frequencies, we can set $v(\omega) \sim (\beta\omega)^{-1}$ and consequently

$$T \sum_\omega G_0^2(p, \omega) \simeq -S^2/p^4$$

At last one gets for Ψ an expression coinciding with a similar expression in the static case,

$$\Psi = -\frac{1}{4} \frac{\partial}{\partial \Lambda^2} \int_p G_0^2(p) \Big|_{\Lambda = 1} \tag{4.11}$$

It should be noted that, taking account of Eq. (4.11), the expression (4.3) for g_{20} coincides with the value of this vertex at the stable fixed point of the conventional approach to the static RG, which appears in the φ^4 theory in the calculation of diagrams like that presented in Fig. 1a.[9,10] In fact, with infinitesimal elimination of short-wavelength modes and an abrupt stepped cutoff $S=\theta(1-p^2/\Lambda^2)$, an integration is made over a narrow band of momenta in the vicinity of $q/\Lambda=1$, which produces the same result. The coincidence of the integral (4.11) and the value g_{20} with the respective values of the one-loop approximation of the standard theory suggests the possibility to transform the RG equations used here in the first ε approximation into the respective one-loop equations. This transformation is presented in Appendix B.

Let us analyze now the integral $I(q, \omega)$. First it should be rewritten in the coordinate representation

$$I=T^2\sum_{\omega_{1,2}}\int d^4r \iint_0^1 \frac{dx}{x}\, dy\; yh(xyr, \omega_1) \times h(ry, \omega_2)\, h(r, -\omega-\omega_1-\omega_2)\, e^{iqrxy} \tag{4.12}$$

in which all further transformations are a little more compact. Then it is convenient first to consider the second derivative with respect to the vector q:

$$\frac{\partial^2 I}{\partial q^2}=-T^2\sum_{\omega_{1,2}}\int\frac{d^4r}{r^2}h(r, -\omega-\omega_1-\omega_2)\int_0^r d(yr)(yr)\, h(yr\omega_2) \times\int_0^{yr} d(xyr)(xyr)\, h(xyr, \omega_1)\, e^{iqrxy}$$

The convenience of doing this becomes evident after explicitly making the integration over angles $\int d\Omega$ in the d-dimensional space

$$\frac{\partial^2 I}{\partial q^2}=-\frac{T^2}{8}\sum_{\omega_{1,2}}\int d\Omega\int_0^\infty dr^2\, h(r, -\omega-\omega_1-\omega_2) \times\int_0^{r^2} du\, h(u, \omega_2)\int_0^u du'\, h(\omega, u')\exp[iq(u')^{1/2}\cos\theta]$$

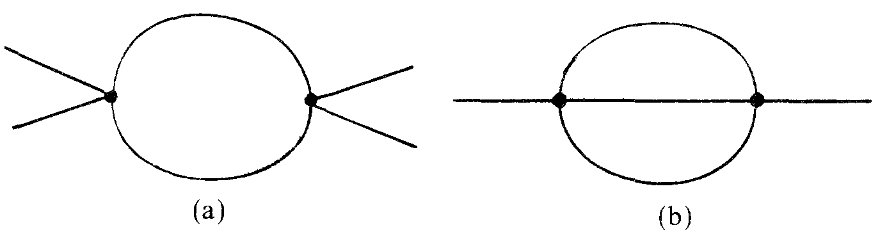

Fig. 1. Perturbation theory diagrams in the φ^4 model which relate to (a) the integral Ψ [Eq. (4.11)]; (b) the integral I [Eq. (4.16)].

and the integration by parts in $\int dr^2$,

$$\frac{\partial^2 I}{\partial q^2}=\frac{T^2}{8}\sum_{\omega_{1,2}}\int d\Omega\int_0^\infty du\left[h(u,\omega_2)\int du\,h(u,\,-\omega-\omega_1-\omega_2)\right]$$

$$\times\int_0^u du'\,h(u',\omega_1)\exp[iq(u')^{1/2}\cos\theta]$$

The integral over u in brackets is indefinite. The shift of the variable of summation ω_2 makes it possible to transform the expression in these brackets into

$$\frac{1}{2}\left[h(u,\omega_2)\int du\,h(u,\,-\omega-\omega_1-\omega_2)+h(u,\,-\omega-\omega_1-\omega_2)\int du\,h(u,\omega_2)\right]$$

$$=-\frac{1}{2}\frac{\partial}{\partial u}\left[\int du\,h(u,\omega_2)\int du\,h(u,\,-\omega-\omega_1-\omega_2)\right]$$

and perform one more integration by parts in the equation for

$$\frac{\partial^2 I}{\partial q^2}=-\frac{T^2}{16}\sum_{\omega_{1,2}}\int d\Omega\int_0^\infty du$$

$$\times\,h(u,\omega_1)\exp[iq(u')^{1/2}\cos\theta]\int h(u,\omega_2)\,du$$

$$\times\int h(u,\,-\omega-\omega_1-\omega_2)\,du$$

$$=-\frac{T^2}{8}\sum_{\omega_{1,2}}\int\frac{d^4r}{r^2}h(r,\omega_1)\int dr^2$$

$$\times\,h(r,\omega_2)\int dr^2\,h(r,\,-\omega-\omega_1-\omega_2) \qquad (4.13)$$

Then it is necessary to investigate the structure of the integral $\int dr^2\,h(r,\omega)$. For this purpose let us use the general property of the function $h(q,\omega)$ represented by (4.11), $h=-\partial(q^2G_0)/\partial q^2$, and rewrite $\int dr^2\,h(r,\omega)$ as

$$\int dr^2\,h(r,\omega)=-\frac{1}{2}\int\frac{dr^2}{r^2}\int d\Omega\int_0^\infty dq^2\,(rq)^2\,e^{iqr\cos\theta}\frac{\partial(q^2G_0)}{\partial q^2}$$

$$=\frac{1}{2}\int d\Omega\int\frac{dr^2}{r^2}\int_0^\infty dq^2\,q^2G_0\frac{\partial}{\partial q^2}(rq)^2\,e^{iqr\cos\theta}$$

Provided that $x\,\partial f(xy)/\partial x = y\,\partial f(xy)/\partial y$, one gets

$$\int dr^2\, h(r,\omega) = r^2 \int d^4q\, G_0(q,\omega)\, e^{iqr} = r^2 G_0(r,\omega) \tag{4.14}$$

Substitution of this result into Eq. (4.13) immediately solves the problem of analysis of $I(q, \omega)$:

$$\frac{\partial^2 I}{\partial q^2} = \frac{\partial}{\partial q^2}\left(\frac{T^2}{8}\sum_{\omega_{1,2}}\int d^4r\, e^{iqr} h(r,\omega_1)\, G_0(r,\omega_2)\, G_0(r, -\omega-\omega_1-\omega_2)\right)$$

Hence, within an accuracy of $I\,|_{q=0}$ one has

$$\begin{aligned} I &= \frac{T^2}{8}\sum_{\omega_{1,2}}\int d^4r\, e^{iqr} h(r,\omega_1)\, G_0(r,\omega_2)\, G_0(r, -\omega-\omega_1-\omega_2) \\ &= \frac{1}{24}\frac{\partial}{\partial \Lambda^2} T^2 \sum_{\omega_{1,2}}\int d^4r\, e^{iqr} G_0(r,\omega_1) \\ &\quad \times G_0(r,\omega_2)\, G_0(r, -\omega-\omega_1-\omega_2)\Big|_{\Lambda=1} \end{aligned} \tag{4.15}$$

Our further procedure involves the routine calculation of sums over imaginary frequencies $\omega_{1,2}$, which can be performed with the help of the spectral representation for $G_0(q, \omega)$,

$$G_0(q,\omega) = \int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \frac{\operatorname{Im} G_0(q,\Omega)}{\Omega - \omega - i\delta}$$

where $\delta = +0$. Bearing in mind that the functions $v(\omega)$ appearing in the summation can be represented, just as before, as $v \sim T/\omega$, one gets

$$\begin{aligned} I(q,\omega) = 12 \int_{k,p} \iint_{-\infty}^{\infty} & \frac{d\Omega_1\, d\Omega_2}{\pi^2} \frac{\operatorname{Im} G_0(k,\Omega_1)}{\Omega_1} \\ & \times \frac{\operatorname{Im} G_0(p,\Omega_2)}{\Omega_2} G_0(\rho, -\Omega_1-\Omega_2-\omega) \end{aligned} \tag{4.16}$$

(where $\rho = k + p + q$). Expression (4.16) coincides with a similar integral of ref. 9. In the diagram technique, this integral has the corresponding diagram presented in Fig. 1b. Performing an integration over Ω_1 and Ω_2 in Eq. (4.16), taking account of the explicit form of the function $G_0(q, \omega) = S(q)/(q^2 + \omega\varphi)$, one gets

$$I(q,\omega) = \int_{k,p} \frac{S(k)\, S(p)\, S(\rho)}{(kp\rho)^2} \frac{(k^2+p^2+\rho^2)}{k^2+p^2+\rho^2-\omega\varphi} \tag{4.17}$$

For the abrupt cutoff $S(q) = \theta(1 - q^2/\Lambda^2)$ and $\varphi = \Gamma^{-1}$, (4.17) coincides with the corresponding expression given in ref. 20.

The limit of importance is $\omega \to 0$. In this case, Eq. (4.17) obviously gives the static result for $I(q, 0)$: $I(q, 0) = \int_{k,p} S(k)\, S(p)\, S(\rho)/(kp\rho)^2$. It can be easily proved that the respective integral $I(q, 0)$ in the static limit has the form

$$I(q, 0) = \int_{k,p} h(k)\, h(p) \iint_0^1 \frac{dx}{x}\, dy\; y h([(q+k)x + p]\, y) \tag{4.18}$$

where $h(q) \equiv h(q, \omega = 0)$. Using the integral (4.18), one can calculate the Fisher exponent determined by Eq. (3.7),

$$\eta \equiv \eta(0) = 6(n+2)\, g_{20}^2 \int_{k,p} h(k)\, h(p) \int_0^1 dx\, x \frac{\partial h(p+kx)}{\partial (p+kx)^2} \tag{4.19}$$

Equation (4.19) looks rather cumbersome and casts some doubt as to the universality of the obtained exponent. It has relatively arbitrary functions $h(q)$ in the integrands. It can be shown, however, that the final result does not depend on the particular choice of the cutoff function $S(q)$ and coincides with the known magnitude $\eta = (n+2)/2(n+8)^2$. The proofs of this statement available in the literature are rather complicated.[21–24] In Appendix C a new proof based on the results obtained in this section is given. It also should be emphasized that the proof of the universality of integral (4.19) leaves the question open about the cause of the coincidence of η in different approaches and about the problem of universality of the corresponding integrals in the standard approach upon the replacement of an abrupt cutoff with an arbitrary and smooth one. Reducing the integrals to each other solves this problem automatically. The same is also valid for the integral Ψ, also considered in Appendix C.

The coincidence of the integral (4.16) with the similar expression of the standard approach[9,10] makes it possible, in turn, to use the expressions for $\Gamma(\omega)$ and the related dynamical exponent z obtained within the standard approach. Namely: $\Gamma(\omega) = \Gamma[1 + 3g_{20}^2(n+2)\ln(4/3)/8\pi^4 \ln(i\omega)]$ and $z = 2 + [6\ln(4/3) - 1]\eta$ for the case of (2.3a); and $\Gamma = \text{const}$ and $z = 4 - \eta$ for the case of (2.3b). These calculations are performed with the help of the stepped cutoff S and they leave the question open about the universality of the results obtained within the dynamics for a more arbitrary choice of S.

Thus, the approach to the theory of critical phenomena presented here and based on the analysis of the scale equations for the correlation functions can be in equal measure successfully applied to the investigation of both static and dynamic properties of a substance in the vicinity of the phase transition point. The established close relationship between this approach and other versions of the theory of critical phenomena (namely, between the Wilson exact functional RG on the one hand and the perturbation theory in the φ^4 model on the other) provides much freedom in selecting a calculation technique. Thus, it allows one to choose the technique most suitable (from the point of view of simplicity and mathematical convenience) for the solution of some particular task.

APPENDIX A. FIXED POINT OF THE RG EQUATIONS IN THE SECOND ORDER IN ϵ

The fixed point of the RG equations in the second order in ε is determined by the system of equations

$$U_k[g_j\{q_i, \omega_i\}] = 0 \tag{A1}$$

possessing an appropriate number of vertices g_k. It can be easily checked that at the fixed (nontrivial) point the vertices g_{10} and g_2 have the order ε, whereas $g_3 \sim \varepsilon^2$ and other vertices $g_{k>3}$ are of higher order. Therefore, to an accuracy of ε^2 it is sufficient to consider the following three equations:

$$g_{10} = -\frac{1}{2}(n+2)\, g_{20} T \sum_{\omega_p} \int_p h(p, \omega_p) \tag{A2}$$

$$\left\{\varepsilon - R_2\{q_i\} - 2g_{10} \sum_{i=1}^{2} [h(q_i, \omega_i) + (i \leftrightarrow \bar{i})]\right\} \times g_2\{q_i, q_{\bar{i}}; \omega_i, \omega_{\bar{i}}\} + Q_2\{q_i, q_{\bar{i}}; \omega_i, \omega_{\bar{i}}\} = 0 \tag{A3}$$

$$[2 + R_3\{q_i\}]\, g_3\{q_i, q_{\bar{i}}; \omega_i, \omega_{\bar{i}}\} + T \sum_{\omega_p} \int_p h(p)\, \hat{S}[g_2(p, \{q_i\}; \omega_p, \{\omega_i\})\, g_2(-p, \{q_i\}; -\omega_p, \{\omega_i\})] = 0 \tag{A4}$$

Here the function $Q_2\{q_i, \omega_i\}$ is determined by Eq. (24),

$$R_k\{q_i\} = \sum_{i=1}^{k} \left(q_i \frac{\partial}{\partial q_i} + q_{\bar{i}} \frac{\partial}{\partial q_{\bar{i}}} \right)$$

Equation (A2) is written to an accuracy of order ε, since $\eta(q, \omega) \sim g_{10}^2$. For the following solution it is convenient to distinguish explicitly the constant part of the vertex $g_2 = g_{20} + \bar{g}_2$ in Eq. (A3). It is determined by the equation

$$\varepsilon g_{20} - 8 g_{10}\, g_{20} h(0, 0) + Q_2\{q_i = 0, \omega_i = 0\} = 0 \tag{A5}$$

The value g_{20} is of order ε, while the (q, ω)-dependent part, as can be verified, will be of the next order. The equation for $\bar{g}_2$ has the form

$$R_2\{q_i\}\ \bar{g}_2(q_i, \omega_i) = -[\chi\{q_i, \omega_i\} - \chi\{0\}] \tag{A6}$$

where

$$\chi\{q_i, \omega_i\} = 2g_{10}\, g_{20} \sum_{i=1}^{2} [h(q_i, \omega_i) + h(q_i, \omega_{\bar{i}})] - Q_2 \tag{A7}$$

Retaining in Eq. (A4) terms of the order of $g_{20}^2 \sim \varepsilon^2$, one obtains the considerably simpler relation

$$[2 + R_3\{q_i\}]\ g_3\{q_i, \omega_i\} = -g_{20}^2 H\{q_i, \omega_i\} \tag{A8}$$

where

$$H\{q_i, \omega_i\} = \frac{2}{3} \sum_{i \neq j}^{3} [h(q_i + q_j + q_{\bar{j}}, \omega_i + \omega_j + \omega_{\bar{j}}) + (i \to \bar{i})]$$

Solutions of Eqs. (A6) and (A8) can be formally written as

$$\bar{g}_2 = -\int_0^1 \frac{dx}{x} [\chi\{q_i x, \omega_i\} - \chi\{0, 0\}] \tag{A9}$$

$$g_3\{q_i, \omega_i\} = -g_{20}^2 \int_0^1 dy\ yH\{yq_i, \omega_i\} \tag{A10}$$

The function (A10) determines the function $Q_2\{q_i, \omega_i\}$ which in turn [see Eq. (A7)] determines $\bar{g}_2\{q_i, \omega_i\}$. Thus, one can easily find that

$$Q_2\{0, 0\} = -4g_{20}^2 \sum_{\omega_p} \int_p h(p, \omega_p) \int_0^1 dy\ y[(n+8)\, h(py, \omega_p) - 2(n+2)\, h(0)] \tag{A11}$$

and, consequently, according to Eq. (A5) one gets

$$g_{20} = \varepsilon \left[4(n+8) \sum_{\omega_p} \int_p h(p, \omega_p) \int_0^1 dy\ yh(py, \omega_p) \right]^{-1} \tag{A12}$$

The function $D(q, \omega)$ necessary to find contributions to $\dot{g}_1(q, \omega)$ is determined through the difference

$$D(q, \omega) = Q_1(q, \omega) - g_{10}^2 h(q, \omega) \tag{A13}$$

Using Eqs. (A9) and (A11), one gets

$$
\begin{aligned}
Q_1(q,\omega) = & -4g_{10}\,g_{20}(n+2)\,T\sum_{\omega}\int_0^1 \frac{dx}{x}\,h(qx,\omega)\,T\sum_{\omega_p}\int_p h(p,\omega) \\
& -4g_{20}^2T^2\sum_{\omega_{k,p}}\int_{k,p} h(k,\omega_k)\,h(p,\omega_p)\iint_0^1 \frac{dx}{x}\,dy\,y \\
& \times\{(n+2)^2\,h(qxy,\omega)+6(n+2) \\
& \times h([p+(k+q)x]\,y,\,-\omega-\omega_k-\omega_p)\}-(k,q=0)
\end{aligned}
\tag{A14}
$$

Substituting (A14) into (A13), one finally has

$$
\begin{aligned}
D(q,\omega) = {} & 24(n+2)\,g_{20}^2T^2\sum_{\omega_{k,p}}\int_{k,p} h(k,\omega_k)\,h(p,\omega_p) \\
& \times\iint_0^1 \frac{dx}{x}\,dy\;yh([p+(k+q)x]\,y,\,-\omega-\omega_k-\omega_p)-(k,q=0)
\end{aligned}
\tag{A15}
$$

The obtained expression completely determines the sought dependence of $\dot{g}_1$ on variables q and ω.

APPENDIX B. CONNECTION OF EXACT RG EQUATIONS WITH THE STANDARD PERTURBATION THEORY (ONE-LOOP APPROXIMATION)

In Section 4 it was established that the integral Ψ [see Eq. (31)] and, consequently, the expression for the vertex g_{20} coincide with those in the one-loop approximation of the conventional RG approach. So it seems quite natural to try to reduce the RG equations from different approaches to each other. In the lowest order in ε this will be done in this Appendix. For the sake of brevity, the summation over ω, and the ω dependence of fields $\varphi(q,\omega)$ and values $h(q,\omega)$ on this variable, will be omitted. This is fully justified, since, first, the proper RG transformations in dynamics and statics coincide, and, second, formally the frequency dependence is restored by means of replacements

$$
\int_q \to T\sum_{\omega}\int_q; \qquad \varphi(q)\to\varphi(q,\omega); \qquad h(q)\to h(q,\omega)
$$

In the lowest ε order the value η in the RG equations can be omitted. Equations (2.11) are generated by Eq. (2.12) for variatonal derivatives for $\mathscr{H}_I$

$$\dot{\mathscr{H}}_I = \int_q \left[(\hat{\varepsilon}_q \varphi(q)) \frac{\delta \mathscr{H}_I}{\delta \varphi(q)} + h(q) \left(\frac{\delta^2 \mathscr{H}_I}{\delta \varphi(q)\, \delta \varphi(-q)} - \frac{\delta \mathscr{H}_I}{\delta \varphi(q)} \frac{\delta \mathscr{H}_I}{\delta \varphi(-q)} \right) \right] \tag{B1}$$

where the operator $(d+2)/2 + q\, \partial/\partial q$ is denoted, for brevity, by $\hat{\varepsilon}_q$.

The main idea of the further transformations consists in eliminating the last term in (B1) giving rise to contributions to the higher order vertices ($\hat{g}_k$ with a higher number k) by the low-order ones. For this purpose let us pass now to a new functional $\bar{\mathscr{H}}_I$ using the transformation

$$\bar{\mathscr{H}}_I = \mathscr{H}_I - \frac{1}{2} \int_q a(q) \frac{\delta \mathscr{H}_I}{\delta \varphi(q)} \frac{\delta \mathscr{H}_I}{\delta \varphi(-q)} \tag{B2}$$

where the function $a(q)$ will be specially adjusted. The functional $\bar{\mathscr{H}}_I$ will be assumed to have smallness in ε and we retain only terms to the second order in $\bar{\mathscr{H}}_I$ appearing after the substitution of (B2) into (B1), i.e.,

$$\begin{aligned} \dot{\bar{\mathscr{H}}}_I \simeq \int_q \Bigg\{ & (\hat{\varepsilon}_q \varphi(q)) \frac{\delta \bar{\mathscr{H}}_I}{\delta \varphi(q)} + h(q) \left(\frac{\delta^2 \bar{\mathscr{H}}_I}{\delta \varphi(q)\, \delta \varphi(-q)} - \frac{\delta \bar{\mathscr{H}}_I}{\delta \varphi(q)} \frac{\delta \bar{\mathscr{H}}_I}{\delta \varphi(-q)} \right) \\ & + \left[(\hat{\varepsilon}_q \varphi(q)) \frac{\delta}{\delta \varphi(q)} + h(q) \frac{\delta^2}{\delta \varphi(q)\, \delta \varphi(-q)} \right] \\ & \times \int_p \frac{a(p)}{2} \frac{\delta \mathscr{H}_I}{\delta \varphi(p)} \frac{\delta \mathscr{H}_I}{\delta \varphi(-p)} - a(q) \frac{\delta \bar{\mathscr{H}}_I}{\delta \varphi(-q)} \frac{\delta}{\delta \varphi(q)} \\ & \times \int_q \left[(\hat{\varepsilon}_p \varphi(p)) \frac{\delta \mathscr{H}_I}{\delta \varphi(p)} + h(p) \frac{\delta^2 \mathscr{H}_I}{\delta \varphi(p)\, \delta \varphi(-p)} \right] \Bigg\} \end{aligned} \tag{B3}$$

Now let us dispose of the function $a(q)$ so as to eliminate all terms of the type $[\delta \bar{\mathscr{H}}_I / \delta \varphi(q)]\ \delta \bar{\mathscr{H}}_I / \delta \varphi(-q)$. It can be easily proved that for this purpose $a(q)$ should satisfy the equation $(1 + q^2\, d/dq^2)\, a(q) = -h(q)$. It should be noted that both in statics and in dynamics the function $h(q)$ [or $h(q, \omega)$] is connected with G_0 via the relation (4.10), $h = -\partial(q^2 G_0)/\partial q^2$. The equation for $a(q)$ has an obvious integral $a(q) = G_0(q)$. After elementary transformations, Eq. (B3) acquires the form

$$\begin{aligned} \dot{\bar{\mathscr{H}}} = \int_q \Bigg\{ & (\hat{\varepsilon}_q \varphi(q)) \frac{\delta}{\delta \varphi(q)} + h(q) \frac{\delta^2}{\delta \varphi(q)\, \delta \varphi(-q)} \Bigg] \bar{\mathscr{H}}_I \\ & + \int_p h(q)\, G_0(p) \frac{\delta^2 \bar{\mathscr{H}}_I}{\delta \varphi(q)\, \delta \varphi(p)} \frac{\delta^2 \bar{\mathscr{H}}_I}{\delta \varphi(-q)\, \delta \varphi(-p)} \Bigg\} \end{aligned} \tag{B4}$$

Finally, since $h(q)=\Lambda^2\,\partial G_0/\partial\Lambda^2|_{\Lambda=1}$, one gets the sought equation of the first ε approximation,

$$\dot{\mathscr{H}}_I=\int_q(\hat{\varepsilon}_q\varphi(q))\frac{\delta\mathscr{H}_I}{\delta\varphi(q)}+\frac{\partial}{\partial\Lambda^2}\int_q G_0(q)\\ \times\left[\frac{\delta^2\mathscr{H}_I}{\delta\varphi(q)\,\delta\varphi(-q)}+\int_p\frac{G_0(p)}{2}\frac{\delta^2\mathscr{H}_I}{\delta\varphi(q)\,\delta\varphi(p)}\frac{\delta^2\mathscr{H}_I}{\delta\varphi(-q)\,\delta\varphi(-p)}\right]\Bigg|_{\Lambda=1} \tag{B5}$$

In the φ^4 model this equation coincides with the one-loop equation appearing in the calculation of the diagram presented in Fig. 1. As mentioned above, the ω dependence in Eq. (B5) is easily restored by means of the formal replacement $\int_q\to T\sum_\omega\int_q$, $\varphi(q)\to\varphi(q,\omega)$, $G(q)\to G(q,\omega)$. Thus, the integral Ψ containing $\sum_\omega$ appears in the right-hand side. After the summation [see (4.11)] this integral fully coincides with the similar integral for the RG in statics.

APPENDIX C. PROVING THE UNIVERSALITY OF INTEGRALS Ψ AND $(\partial^2 I/\partial q^2)|_{\omega=q=0}$

One of the main hypotheses in the theory of critical phenomena is the statement that the critical asymptotics to do not depend on the cutoff method, i.e., on the choice of a particular function $S(q)$. This statement is often used to justify the fact that the calculations are carried out with some special form of $S(q)$ in particular $S=\theta(1-q^2/\Lambda^2)$. Strictly speaking, however, it is a hypothesis yet and it still must be proved as to the universality of each obtained result. In this Appendix this hypothesis will be proved with respect to the integrals Ψ and $\partial^2 I/\partial q^2|_{q=\omega=0}$ involved in the determination of the exponent η. Similar proofs were specially dealt with in refs. 21–24. It seems to us that the proof given below in the context of this paper is more obvious and compact. Unfortunately, we could not get a similar proof of the universal character of the index z.

Let us start with Ψ. Using properties of the functions $h(q,\omega)=-d(G_0q^2)/dq^2$, $S(0)=1$, and $S(q\to\infty)\to 0$, one gets

$$\Psi=-\frac{T}{2}\sum_\omega\int_p h(p,\omega)\,G_0(p,\omega)\\ =\frac{K_4}{4}T\sum_\omega\int_0^\infty dp^2\,\frac{d(G_0p^2)}{dp^2}(G_0p^2)\\ =\frac{K_4}{8}\left\{\sum_\omega[p^2G_0(p,\omega)]^2\right\}\Bigg|_0^\infty=\frac{K_4}{8}S(0)=\frac{K_4}{8} \tag{C1}$$

where $K_d = S_d/(2\pi)^d$; S_d is the area of the unit-radius d-dimensional sphere. Thus, the integral Ψ is obviously universal.

For $\partial^2 I/\partial q^2|_{q=\omega=0}$ it is useful to exploit the representation

$$-\frac{\partial^2 I}{\partial q^2}\bigg|_{q=\omega=0} = \frac{1}{8}\int d^4r\, r^2 h(r)\, G_0^2(r) \tag{C2}$$

Using now Eq. (4.14) and denoting $\kappa(r^2) = G_0 r^2$, one has

$$\frac{\partial^2 I}{\partial q^2}\bigg|_{q=\omega=0} = \frac{1}{16}\int d\Omega \int_0^\infty dr^2\, \kappa^2(r^2)\, \frac{d\kappa}{dr^2} = \frac{S_4}{3\cdot 16}\,(4\kappa(0))^3 \tag{C3}$$

Now the only thing is to calculate $\kappa(0)$, given by

$$\kappa(0) = \int_p \frac{d^2h}{dp^2} = \frac{K_4}{2}\int_0^\infty dp^2\, \frac{dS}{dp^2} = \frac{K_4}{2}\, S(0) = \frac{K_4}{2} \tag{C4}$$

which directly proves the desired universality.

APPENDIX D

This Appendix is intended to derive the "classical" Ginzburg–Landau functional from the microscopic Hamiltonian of a nonideal Bose gas, which is

$$\hat{H} = \hat{H}_0 + \hat{H}_I \tag{D1}$$

$$\hat{H}_0 = \int dr\, \hat{\psi}^+(r)\left(-\frac{\nabla^2}{2m}\right)\hat{\psi}(r) \tag{D2}$$

$$\hat{H}_I = \frac{1}{2}\int dr\, dr'\, \hat{\rho}(r)\, V(r-r')\, \hat{\rho}(r') \tag{D3}$$

where $\hat{\psi}^+$ and ψ are the field Bose operators, $\hat{\rho} = \hat{\psi}^+\hat{\psi}$ is the particle density operator, and the potential $V(r)$, for the sake of simplicity, will be regarded as purely repulsive, i.e., $V(r) > 0$ (the Ginzburg–Landau functional for the classical gas in which, along with short-range repulsion, there is long-range attraction was constructed in refs. 25 and 26).

Our task is to calculate (to be more exact, to give the representation in the form of the continuum integral over the c-quantity field) of the partition function of the grand canonical ensemble

$$Z = \mathrm{Sp}\, \exp[-\beta(\hat{\hat{H}}_0 + \hat{H}_I)] \tag{D4}$$

where $\hat{\bar{H}}_0 = \hat{H}_0 - \mu \hat{N}$, μ is the chemical potential, and $\hat{N} = \int dr\, \hat{\rho}(r)$ is the operator of the total number of particles in the system. The partition function (A4) can be represented in the form[27]

$$Z = \mathrm{Sp}\{[\exp(-\beta \hat{\bar{H}}_0)]\, \hat{\sigma}(\beta)\} \tag{D5}$$

where

$$\hat{\sigma}(\beta) = \hat{T}_\tau \exp\left[-\frac{1}{2} \int_0^\beta d\tau \int dr\, dr'\, \hat{\rho}(r, \tau)\, V(r - r')\, \hat{\rho}(r', \tau) \right] \tag{D6}$$

Here $\hat{T}_\tau$ is the ordering operator along the imaginary time axis, $0 \leqslant -i\tau \leqslant -i\beta$, and the time dependence of operators is taken in the interaction representation.

Using the Stratonovich–Habbard transformation, one can represent the operator $\hat{\sigma}(\beta)$ as a functional quadrature over the c-quantity field $\theta(r, \tau)$,

$$\begin{aligned} \hat{\sigma}(\beta) = A \int D\theta \exp\left[-\frac{1}{2} \int_0^\beta d\tau \int dr\, dr'\, \theta(r, \tau)\, K(r - r')\, \theta(r', \tau) \right] T_\tau \\ \times \exp\left[-i\beta \int dr\, \theta(r, \tau)\, \hat{\rho}(r, \tau) \right] \end{aligned} \tag{D7}$$

where

$$A^{-1} = \int D\theta(r, \tau) \exp\left[-\frac{1}{2} \int_0^\beta d\tau \int dr\, dr'\, \theta(r, \tau)\, K(r - r')\, \theta(r', \tau) \right]$$

and the function $K(r - r')$ is determined by

$$\int d\bar{r}\, V(r - \bar{r})\, K(\bar{r} - r') = \delta(r - r') \tag{D8}$$

With the help of representation (D7), the partition function (D6) can be written in the form

$$Z = A \int D\theta \exp\{ -\beta \mathscr{H}[\theta] \} \tag{D9}$$

$$\begin{aligned} \beta \mathscr{H}[\theta] = \frac{1}{2} \int_0^\beta d\tau \int dr\, dr'\, \theta(r, \tau)\, K(r - r')\, \theta(r', \tau) \\ - \ln \mathrm{Sp} \Big\{ [\exp(-\beta \hat{\bar{H}}_0)]\, T_\tau \\ \times \exp\left[-i \int_0^\beta d\tau \int dr\, \theta(r, \tau)\, \hat{\rho}(r, \tau) \right] \Big\} \end{aligned} \tag{D10}$$

In the mean field theory the equilibrium value $\theta(r, \tau)$ corresponds to the minimum of the functional (D10) and is determined by

$$\begin{aligned}\beta \frac{\delta \mathscr{H}}{\delta\theta(r,\tau)} = 0 = \int dr'\, K(r-r')\,\theta(r',\tau) \\ + i\left(\mathrm{Sp}\left\{[\exp(-\beta\hat{\bar{H}}_0)]\,\hat{T}_\tau \hat{\rho}(r,\tau)\right.\right. \\ \left.\left. \times \exp\left[-i\int_0^\beta d\tau \int dr'\,\rho(r',\tau)\,\theta(r',\tau)\right]\right\}\right) \\ \times\left(\mathrm{Sp}\left\{[\exp(-\beta\hat{\bar{H}}_0)]\,\hat{T}_\tau\right.\right. \\ \left.\left. \times \exp\left[-i\int_0^\beta d\tau \int dr'\,\rho(r',\tau)\,\theta(r',\tau)\right]\right\}\right)^{-1}\end{aligned} \tag{D11}$$

The space-homogeneous and time-independent solution of Eq. (D11) has the form

$$\theta_0 = -i\left[\int dr\, K(r)\right]^{-1} \langle\!\langle \hat{\rho} \rangle\!\rangle_0 \tag{D12}$$

where $\langle\!\langle \cdots \rangle\!\rangle$ denotes the quantum mechanical averaging with Hamiltonian $\hat{\bar{H}}_0 + i\theta_0 \hat{N}$. It is evident that the value $\langle\!\langle \rho \rangle\!\rangle_0$ is the density of a noninteracting Bose gas with chemical potential $\mu + i\theta_0$: $\langle\!\langle \rho \rangle\!\rangle_0 = n(\mu + i\theta_0 T)$.

The approximation $\Omega_0 = \mathscr{H}[\theta_0]$ determines the thermodynamic potential of the interacting Bose gas in the mean field approximation. The gas density in this approximation is

$$n_0(\mu, T) = \frac{1}{V}\left(-\frac{\partial \Omega_0}{\partial \mu}\right)_T = \frac{1}{V}\langle\!\langle \hat{N} \rangle\!\rangle_0 = n(\mu + i\theta_0, T) \tag{D13}$$

With the account of relation (D13) the system pressure is expressed by

$$P = P_0(\mu + i\theta_0, T) = \frac{1}{2}\left[\int dr\, K(r)\right]^{-1} n^2(u + i\theta_0, T) \tag{D14}$$

where $P_0(\mu + i\theta_0, T)$ is the pressure of an ideal Bose gas, and by the chemical potential $\mu + i\theta_0$.

Since our aim is to construct the Ginzburg–Landau functional for the investigation of fluctuation effects, let us set $\theta(r, \tau) = \theta_0 + \varphi(r, \tau)$ and expand the functional $\mathscr{H}[\theta]$ in $\varphi(r, \tau)$:

$$\beta\, \Delta\mathscr{H}[\varphi] = \frac{1}{2}\int_0^\beta d\tau\, d\tau' \int dr\, dr'\, \varphi(r, t)\, \varphi(r', \tau')$$
$$\times\ [\delta(\tau - \tau')\, K(r - r') + \langle\!\langle \hat{T}_\tau \hat{\rho}(r, \tau)\, \hat{\rho}(r', \tau')\rangle\!\rangle_{0c}] + \cdots \qquad \text{(D15)}$$

where $\langle\!\langle \cdots \rangle\!\rangle_{0c}$ is the connected average with Hamiltonian $\hat{\hat{H}}_0 + i\theta_0 \hat{N}$.

The Fourier transform of the expression in square brackets on the right-hand side of Eq. (D15) determines the free propagator $G_0^{-1}(q, \omega)$, which constitutes a part of the functional (2.2). Our next task is to calculate this propagator for the model under consideration. Opening the average $\langle\!\langle \cdots \rangle\!\rangle_{0c}$, one obtains

$$G_0^{-1} = V_q^{-1} + T\sum_{\omega'} \int_{q'} \tilde{G}_0(q', \omega')\, \tilde{G}_0(q' - q, \omega' - \omega) \qquad \text{(D16)}$$

Here V_q is the Fourier transform of the particle interaction potential

$$\tilde{G}_0(q, \omega) = -\int_0^\beta d\tau \int dr\, e^{i(\omega\tau - qr)} \langle\!\langle \hat{T}_\tau \psi(r, \tau)\, \psi^+(0, 0)\rangle\!\rangle_0$$
$$= (i\omega - \varepsilon q)^{-1}$$
$$\varepsilon_q = q^2/2m - \mu + i\theta_0 \qquad \text{(D.17)}$$

and the temperature dependence is restored in the Matsubara frequencies $\omega = 2\pi n\beta i$.

Performing the standard summation over frequencies in Eq. (D16), one gets

$$G_0^{-1}(q, \omega) = \int_q \frac{\nu(\varepsilon_{q'}) - \nu(\varepsilon_{q'-q})}{i\omega - \varepsilon_{q'} + \varepsilon_{q'-q}} \qquad \text{(D18)}$$

where $\nu(\varepsilon)$ is the Bose–Einstein distribution function. As we consider a non-degenerate Bose gas, the replacement of $\nu(\varepsilon)$ with the Boltzmann distribution function in the limit $q \to 0$, $\omega \to 0$ gives

$$G_0^{-1}(q, \omega) = V_q^{-1} - \frac{\Lambda_T^3}{\Gamma V_0}\left[2T - \left(\frac{q}{\Lambda_T}\right)^2 T - i\omega\right] \qquad \text{(D19)}$$

where $\Lambda_T = (2m/T)^{3/2}$ is the thermal momentum, and $\Gamma^{-1} = V_0 \Lambda e^{\beta(\mu + i\theta_0)}/ 6\pi^{3/2}T^2$. Expanding now V_q in the vicinity of $q = 0$ and replacing $\varphi(\omega, q) \to \varphi(q, \omega)\, V_0^{1/2}$ near the transition points, one finally gets

$$\beta\, \Delta\mathscr{H}[\varphi] = \frac{T}{2}\sum_\omega \int_q |\varphi(q, \omega)|^2\, (\tau + cq^2 + i\omega\Lambda_T^2\Gamma^{-1}) + \cdots \qquad \text{(D20)}$$

where $\tau = (T - T_{c0})/T_{c0}$, and $T_{c0} = 2\Gamma^{-1}$ is the trial critical temperature; here

$$c = \frac{1}{2V_0} \frac{\partial^2 V}{\partial q^2}\bigg|_{q=0} + \Lambda_T T_{c0} \Gamma^{-1}$$

Thus, for the transition under consideration the free propagator in the form of (2.3a) has been obtained from microscopics.

The SE are obtained for the correlation functions of the field $\varphi(q, \omega)$, and to investigate the dynamic phenomena one has to calculate the linear response function. The latter is the analytical extension from discrete frequencies to the real axis of the Matsubara temperature Green function. To conclude this Appendix, let us show that the Matsubara Green function is linearly connected with the correlation function used in the SE. For this purpose the Matsubara two-particle function is represented as

$$\begin{aligned}
&\tilde{G}(r, \tau; r', \tau') \\
&\quad = \frac{1}{Z} \operatorname{Sp}\{[\exp(-\beta \hat{\tilde{H}}_0)]\, \hat{T}_\tau \hat{\rho}(r, \tau)\, \hat{\rho}(r', \tau')\, \hat{\sigma}(\beta)\} \\
&\quad = -\frac{1}{Z} \operatorname{Sp}\left\{\exp(-\beta \hat{H}_0) \int D\theta(r, \tau)\right. \\
&\quad\quad \times \exp\left[-\frac{1}{2}\int_0^\beta d\tau \int dr\, dr'\, \theta(r, \tau)\, K(r - r')\, \theta(r')\right] \frac{\delta^2}{\delta\theta(r, \tau)\, \delta\theta(r', \tau')} \\
&\quad\quad \left. \times \hat{T}_\tau \exp\left[-i \int_0^\beta d\tau \int dr\, \theta(r, \tau)\, \hat{\rho}(r, \tau)\right]\right\}
\end{aligned}$$

Performing elementary actions, one gets

$$\begin{aligned}
&\tilde{G}(r, \tau; r', \tau') \\
&\quad = -\left[\int dr\, K(r)\right]^2 \theta_0^2 - \int d\bar{r}\, d\bar{r}'\, K(r - \bar{r})\, K(r' - \bar{r}') \langle \varphi(\bar{r}, \tau)\, \varphi(\bar{r}', \tau') \rangle
\end{aligned} \tag{D21}$$

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